Assignment # 3	Due on November 13th in class

1. Prove that $\mathbb{Q}$ is the accumulating set (the set of all accumulating points) of $\mathbb{R} - \mathbb{Q}$.

2. In a metric space (X, d), for any subset D of X, prove that $\overline{\overline{D}} = \overline{D}$.

3. Prove that l^∞ is not separable.

Hint: First of all, note that $l^\infty = \{(x_1, x_2, \cdots) \colon \sup_{i \in \mathbb{N}_{\geq 1}} |x_i| < \infty\}$. Define $E = \{(x_1, x_2, \cdots) \colon |x_i| \in \{1, 0\}\}$. It is then clear that $E \subset l^\infty$, E is uncountable and for every pair of elements $y_1, y_2 \in D$ with $y_1 \neq y_2$, we have $\text{dist}(y_1, y_2) = 1$. With this in mind, if l^∞ is separable, then there exists $\{z_1, z_2, \cdots\} \subset l^\infty$, such that $\{z_1, z_2, \cdots\}$ is dense in l^∞. You can easily reach an contradiction if you can prove the following:
" we can find y_1 and y_2 in E such that they are both in $B(z_N; 1/3)$ for certain $N \in \mathbb{N}$. "

Solution:

1.

Proof. We just need to show that for any $x \in \mathbb{Q}$ and $\epsilon > 0$, there exists $y \in \mathbb{R} - \mathbb{Q}$, such that $|x - y| < \epsilon$.

For any $\epsilon > 0$, there exists $n \in \mathbb{N}_{\geq 1}$, such that $\frac{\sqrt{2}}{n} < \epsilon$. It then follows that

$$|x - (x + \frac{\sqrt{2}}{n})| < \epsilon.$$

As x is a rational number, and $\sqrt{2}$ is an irrational number, it follows that $x + \frac{\sqrt{2}}{n} \in \mathbb{R} - \mathbb{Q}$, which finishes the proof.

□

2.

Proof. As $D \subset \overline{D}$, we have $\overline{D} \subset \overline{\overline{D}}$. We just need to show that $\overline{\overline{D}} \subset \overline{D}$.

For any $x \in \overline{\overline{D}}$, there exists $x_1, x_2, \cdots \in \overline{D}$, such that $\mathrm{d}(x, x_n) \to 0$. For any n, as $x_n \in \overline{D}$, there exists $y_n \in D$, such that $\mathrm{d}(x_n, y_n) < \mathrm{d}(x, x_n)$. It then follows that

$$\mathrm{d}(x, y_n) < 2\mathrm{d}(x, x_n)$$

where $y_1, y_2, \cdots \in D$. As $\mathrm{d}(x, x_n) \to 0$, we have $\mathrm{d}(x, y_n) \to$, which indicates that $x \in \overline{D}$ for all $x \in \overline{\overline{D}}$, done.

□

3.

Proof. We just start from where the hint ends. In other words, we try to find distinct $y_1, y_2 \in E$, such that $y_1, y_2 \in B(z_n; 1/3)$ for some $n \in \mathbb{N}$. If this can be done, according to the triangle inequality, we have $\mathrm{dist}(y_1, y_2) < 2/3$, contradicting the fact that $\mathrm{dist}(y_1, y_2) = 1$.

We will find such distinct y_1 and y_2 using the pigeon hole principle. Assume that such y_1 and y_2 does not exist, then for any $n \in \mathbb{N}_{\geq 1}$, $E \cap B(z_n; 1/3)$ contains at most one point. As $\{z_n\}_{n \in \mathbb{N}_{\geq 1}}$ is dense, we have $l^\infty = \bigcup_{n=1}^\infty B(z_n; 1/3)$, which, when combined with the fact that $E \cap B(z_n; 1/3)$ contains at most one point, indicates that E is at most a countable set, contradicting the fact that E is uncountable. Done.

□